

THE HEAT KERNEL OF THE COMPACTIFIED $D = 11$ SUPERMEMBRANE WITH NON-TRIVIAL WINDING

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ABSTRACT. We study the quantization of the regularized hamiltonian, H , of the compactified $D = 11$ supermembrane with non-trivial winding. By showing that H is a relatively small perturbation of the bosonic hamiltonian, we construct a Dyson series for the heat kernel of H and prove its convergence in the topology of the von Neumann-Schatten classes so that e^{-Ht} is ensured to be of finite trace. The results provided have a natural interpretation in terms of the quantum mechanical model associated to regularizations of compactified supermembranes. In this direction, we discuss the validity of the Feynman path integral description of the heat kernel for $D = 11$ supermembranes and obtain rigorously a matrix Feynman-Kac formula.

1. INTRODUCTION

The $D = 11$ supermembrane with non-trivial central charge (CSNW) was first analyzed in the semiclassical regime in [1] and [2]. A study of the complete theory, including all the interacting terms in the hamiltonian, was performed later in [3, 4, 5] where it was shown that the spectrum of the $SU(N)$ regularized hamiltonian is a discrete set of eigenvalues of finite multiplicity. In order to have a non-trivial central charge when the spatial part of the world volume is compact, the supermembrane must wrap a compact sector of the target space. This is achieved by imposing a topological restriction on the configuration space. These restrictions are naturally satisfied in the context of brane wrapping calibrated submanifolds, [6, 7, 8]. As it turns out, the ground state configuration of the CSNW is directly related to the intersecting brane solutions of $D = 11$ supergravity, [9, 10, 11, 12, 13, 14].

In the present paper we continue our analysis of the quantum hamiltonian of the CSNW. The approach discussed below aims at a better understanding of the rigorous behaviour of the regularized hamiltonian

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of $D = 11$ supermembranes in the limit $N \rightarrow \infty$. An important step towards the visualization of this limit is provided by Theorem 2 below.

Two are the properties of the CSNW that are crucial in allowing a detailed description the spectrum of the $SU(N)$ regularized hamiltonian, H , [4, 3]. One is the fact that H is a small perturbation of the bosonic hamiltonian $H_B = P^2 + V_B$, cf. (10) below. The other is the certainty that the bosonic potential V_B is bounded below by cQ^2 for suitable $c > 0$, see Lemma 1 below. Our main objective in the present paper is to demonstrate how these properties also lead to an explicit description on the heat kernel of H , see Theorem 2 and Corollary 4.

The main contribution is to be found in Section 4. The heat kernel of $P^2 + Q^2$, the harmonic oscillator, is given by Mehler's formula

$$(1) \quad \text{Ker}[e^{-(P^2+Q^2)t}](x, y) = (w_1/\pi)^{N/2} \exp[2w_1(x \cdot y) - w_2(|x|^2 + |y|^2)]$$

where $w_1 = \frac{\lambda}{1-\lambda^2}$, $w_2 = \frac{1+\lambda^2}{2(1-\lambda^2)}$, and $\lambda = \exp[-2t]$. An explicit computation shows that $\text{Ker}[e^{-(P^2+Q^2)t}] \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$. By dilating the spatial variable, it is easy to show that the latter also hold for $K_t^A := \text{Ker}[e^{-(P^2+cQ^2)t}]$. If we now compare $K_t^B := \text{Ker}[e^{-(P^2+V_B)t}]$ with K_t^A through the Feynman-Kac formula, it becomes clear that also K_t^B is square integrable so that the c_0 one-parameter semigroup $e^{-H_B t}$ is Hilbert-Schmidt and hence (by the semigroup property) of finite trace for all $t > 0$. On the other hand, the results of [3, 5] ensure that H realizes as the Schrödinger-type operator $P^2 + V_B \otimes \mathbb{I} + V_F$, acting on $L^2(\mathbb{R}^N; \mathbb{C}^d)$ for suitably large dimensions $N, d \in \mathbb{N}$, where $V_F : \mathbb{R}^N \rightarrow \mathbb{R}^{d \times d}$ is subordinated to $V_B \otimes \mathbb{I}$ in a sense specified below. Here the spinors are vectors in \mathbb{C}^d so that \mathbb{I} denotes the identity matrix of $\mathbb{C}^{d \times d}$. In Theorem 2 below we demonstrate the convergence for all $t > 0$ of the Dyson expansion for e^{-Ht} in the norm of the von Neumann-Schatten classes of index r for a suitable $r > 2$. Therefore, also e^{-Ht} has a finite trace.

The series expansion found for e^{-Ht} has a natural physical interpretation. If we formally consider the Feynman functional integral for the CSNW and perform the gauge fixed $SU(N)$ regularization procedure described in [5], we achieve a Feynman path integral with light cone coordinates denoted by x^+ . Since the full potential, $V_S := V_B \otimes \mathbb{I} + V_F$, is bounded below, this formula represents the kernel of the Schrödinger operator of a quantum mechanical model with a light cone time whose hamiltonian is H . The heat kernel is the analytic continuation from x^+ onto $-ix^+$ of the above Feynman path integral. Thus, it is of physical relevance constructing this heat kernel for finite N and considering

then the limit as $N \rightarrow \infty$. This limit may have an intrinsic relevance in the description of the CSNW.

Section 5.1 is devoted to describing $K_t := \text{Ker}[e^{-Ht}]$ as a Feynman path integral in terms of the standard Wiener measure. Although this description provides some information about the model, in our view, a deeper insight into the quantization of the CSNW is obtain from the abovementioned comparison between the fermionic and the bosonic sectors of the theory. Following this approach, and using the construction of Section 4, in Section 5.2 we obtain an alternative Feynman-Kac formulation. The latter is given in terms of a measure in the space of continuous path constructed *a la* Wiener, but using the kernel of $e^{-H_B t}$, instead of the free heat kernel, for defining the measure of cylinder sets. This procedure follows the approach suggested in [15, p.49] for the scalar case. In conjunction with the analysis of Section 4, this formula is highly relevant in the construction of a Feynman functional integral formula in the limit $N \rightarrow \infty$.

It is interesting to notice that the results of the remarkable papers [16, 17, 18] are somehow in contrast with the present discussion. The regularized hamiltonian of the supermembrane immerse in $D = 11$ Minkowski target space has spectrum equal to the interval $[0, \infty)$, [16]. If we do not compactify with non-trivial wrapping certain directions in the target space, the existence of locally singular configurations, also known as string-like spikes, force the whole regularized hamiltonian to be incompatible with the bosonic term. By virtue of the spectral theorem, the evolution semigroup can not be a compact operator. Furthermore, in this case the hamiltonian is positive, but since the fermionic potential is not subordinated to the bosonic contribution, a Feynman path integral construction of the heat kernel is still an open problem.

In order to keep our discussion accessible as far as functional analytical properties of H is concerned, Section 3 is devoted to reviewing some standard mathematical tools employed in the subsequent parts of this paper. In Section 2 we briefly justify why H is the correct characterization of a quantum mechanical hamiltonian for the CSNW model. We refer to [3, 5, 19, 20] for a more complete account that such a representation is valid. In [3, 5] we quantize the CSNW by first solving the constraint and the gauge fixing condition, and then performing a canonical reduction of the hamiltonian. For the present discussion we quantize the CSNW model by imposing the constrains on the Hilbert space of states. This second procedure yields to the expression $H = P^2 + V_S$ considered here. It is well known that both schemes of quantization are equivalent.

2. THE REGULARIZED HAMILTONIAN OF THE CSNW

Supermembranes [13] are extended objects which live in 11 dimensions and may couple to 11 dimensional supergravity. These objects were originally proposed over a $D = 11$ Minkowski target space as candidates for fundamental objects. When the possibility of non-empty continuous spectrum was discovered in [16], this led to a reinterpretation of the model as a many body theory. The continuous nature of the spectrum is a consequence of two main facts: the presence of singular configurations, known as string like spikes, with zero energy level and the supersymmetry. This property seems to prevail under the compactification of some directions of the target space, [21], although a rigorous proof such as the one given in the case [16] has not yet been constructed in detail. The spectrum of the CSNW seems to have a completely different nature according to the results of [4, 3]. These supermembranes require considering a compact sector of the target space together with a topological condition on the configuration space. In this section we describe the main ingredients of this construction.

When we consider the $D = 11$ supermembrane in the light cone gauge, [17, 18], the potential is given by

$$V(X) = \{X^{\tilde{M}}, X^{\tilde{N}}\}^2, \quad \tilde{M}, \tilde{N} = 1, \dots, 9,$$

where

$$\{X^{\tilde{M}}, X^{\tilde{N}}\} = \frac{\epsilon^{ab}}{\sqrt{W}} \partial_a X^{\tilde{M}} \partial_b X^{\tilde{N}}, \quad a, b = 1, 2.$$

The scalar density \sqrt{W} is in place as a consequence of the partial gauge fixing procedure and we take it to be the induced volume of a minimal immersion introduced below. Here $X^{\tilde{M}}$ are maps from the compact Riemann surface Σ to the target space. In the present discussion we assume that Σ is a torus and the target space is $\mathbb{M}_7 \times S^1 \times S^1$, where \mathbb{M}_7 is the Minkowski space-time of dimension 7. More general target spaces have been considered in [22]. We take X^r , $r = 1, 2$, to be maps from Σ to $S^1 \times S^1$ and X^m , $m = 3, \dots, 9$ maps from Σ to \mathbb{M}_7 .

For the maps $X^r : \Sigma \longrightarrow S^1$ to be well defined, they should satisfy the condition

$$\oint_{C_i} dX^r = m_{ri} \quad r = 1, 2,$$

where C_i is a basis of homology over Σ and m_{ri} are integers that depend on the indices r and i . Furthermore, for the images of Σ under X^r to describe a torus we should also impose the constraint

$$(2) \quad \int_{\sigma} (dX^r \wedge dX^s) \epsilon_{rs} = 2\pi n \neq 0$$

where $n = \det m_{ri}$ for $r = 1, 2$ and $i = 1, 2$. This condition corresponds to having a non-trivial central charge on the supersymmetric algebra of the supermembrane.

Among all maps from Σ to the target space satisfying the topological condition (2), there is one which minimizes the hamiltonian of the CSNW. This minimizer realizes in terms of the basis of harmonic one-form over σ , $d\hat{X}^r$. Any one-form over σ is given by

$$(3) \quad dX^r = m_s^r d\hat{X}^s + \delta^{rs} dA_s \quad r, s = 1, 2$$

where A_s are single-valued over σ and m_s^r are integers. The map X^r defined by (3) satisfies (2), whenever $\det m_s^r = 1$. Moreover, using the residual gauge invariance, the area preserving diffeomorphisms which are not connected to the identity, we may fix $m_s^r = \delta_s^r$. We are then still left with the diffeomorphisms connected to the identity as a gauge invariance of the theory. The transverse coordinates X^m are valued over $D = 7$ Minkowski space, hence these must be single-valued over Σ .

Under the above considerations, the hamiltonian of the CSNW model can now be rewritten in terms of X^m and A^r . The resulting expression may be found in closed form,

$$(4) \quad \begin{aligned} \tilde{H} = & \int_{\Sigma} (1/2) \sqrt{W} [(P_m)^2 + (\Pi_r)^2 + (1/2) W \{X^m, X^n\}^2 + W (\mathcal{D}_r X^m)^2 + \\ & + (1/2) W (\mathcal{F}_{rs})^2] + \int_{\Sigma} [(1/8) \sqrt{W} n^2 - \Lambda (\mathcal{D}_r \Pi_r + \{X^m, P_m\})] + \\ & + \int_{\Sigma} \sqrt{W} [-\bar{\psi} \Gamma_- \Gamma_r \mathcal{D}_r \psi + \bar{\psi} \Gamma_- \Gamma_m \{X^m, \psi\} + \Lambda \{\bar{\psi} \Gamma_-, \psi\}] \end{aligned}$$

where P_m and Π_r are the conjugate momenta to X^m and A_r respectively. Here \mathcal{D}_r and \mathcal{F}_{rs} are the covariant derivative and curvature, respectively, of a symplectic connection [20] constructed from the symplectic structure $\frac{\epsilon^{ab}}{\sqrt{W}}$ introduced by the central charge, where \sqrt{W} is the volume induced by the minimal immersion \hat{X}^r .

By using the fact that the explicit expression of \tilde{H} is given completely in terms of single-valued objects over Σ , one may find the regularization [5], H . Here the non-exact modes that appear in the regularization of the $D = 11$ supermembrane on compact target space described in [21] are not an issue, since those modes may be gauge fixed as a consequence of the topological condition. The residual gauge symmetry, the diffeomorphisms connected to the identity, are described in terms of

the symplectic non-commutative Yang-Mills formulation (4). In the explicit expression, the term $\Lambda(\mathcal{D}_r \Pi_r + \{X^m, P_m\})$ where Λ is a Lagrange multiplier, describes the generator (Gauss Law) of the symmetry.

The regularized hamiltonian realizes as a self-adjoint operator acting on $L^2(\mathbb{R}^N, \mathbb{C}^d)$, where N and d are large integers. In this realization, the regularized bosonic potential is given explicitly by

$$\tilde{V}_B(X, \mathcal{A}) = V_1(X) + V_2(\mathcal{A}) + V_3(X, \mathcal{A})$$

where

$$\begin{aligned} V_1(X) &= 4 \sum_{D,m,n} \left| \kappa \sum_{B,C} \sin \left(\frac{B \times C}{\kappa} \pi \right) X^{Bm} X^{Cn} \delta_{B+C}^D \right|^2, \\ V_2(\mathcal{A}) &= 4 \sum_{r,s,D} \left| \omega^{(D \times V_r)/2} \kappa \sin \left(\frac{V_r \times D}{\kappa} \pi \right) \mathcal{A}_s^D - \omega^{(D \times V_s)/2} \kappa \sin \left(\frac{V_s \times D}{\kappa} \pi \right) \mathcal{A}_r^D \right. \\ &\quad \left. + i \sum_{B,C} \kappa \sin \left(\frac{B \times C}{\kappa} \pi \right) \mathcal{A}_r^B \mathcal{A}_s^C \delta_{B+C}^D \right|^2, \\ V_3(X, \mathcal{A}) &= 2 \sum_{D,r,m} \left| \omega^{(D \times V_r)/2} \kappa \sin \left(\frac{V_r \times D}{\kappa} \pi \right) X^{Dm} \right. \\ &\quad \left. + i \sum_{B,C} \kappa \sin \left(\frac{B \times C}{\kappa} \pi \right) \mathcal{A}_r^B X^{Cm} \delta_{B+C}^D \right|^2. \end{aligned}$$

Here $\omega = e^{2\pi i/\kappa}$, where κ is a large integer and the indices

$$\begin{aligned} B, C, D &\in \{(a_1, a_2) : a_j = 0, \dots, \kappa - 1, (a_1, a_2) \neq (0, 0)\}, \\ m, n &\in \{1, \dots, 7\}, \quad r, s \in \{1, 2\}, \\ V_1 &= (1, 0), V_2 = (0, 1) \quad \text{and} \quad (a_1, a_2) \times (b_1, b_2) = a_1 b_2 - a_2 b_1. \end{aligned}$$

The non-zero components of \mathcal{A}_r are $\mathcal{A}_1^{(a_1, 0)}$ when $a_1 \neq 0$ and $\mathcal{A}_2^{(a_1, a_2)}$ when $a_2 \neq 0$, while the other components may be gauge fixed to zero [5].

The following lemma was originally formulated in [4, Lemmas 1 and 2].

Lemma 1. *\tilde{V}_B only vanishes at the origin and there exist constants $\tilde{a}, M > 0$ such that*

$$(5) \quad \tilde{V}_B(X, \mathcal{A}) + \tilde{a} \geq M(|X|^2 + |\mathcal{A}|^2), \quad (X, \mathcal{A}) \in \mathbb{R}^N.$$

Proof. The gauge fixing conditions ensure the first part of the lemma, cf. [4, Lemma 1]. For the second part, let \mathbb{T} be the unit ball of \mathbb{R}^N .

We write X^{Bm} and A_r^B in polar coordinates as

$$X^{Bm} = R\phi^{Bm}, \mathcal{A}_r^B = R\psi_r^B \quad \text{or} \quad X = R\phi, \mathcal{A} = R\psi$$

where $R \geq 0$, $\phi = (\phi^{Bm})$, $\psi = (\psi_r^B)$ and $(\phi, \psi) \in \mathbb{T}$. A straightforward calculation yields

$$\tilde{V}_B(R\phi, R\psi) = R^4 k_1(\phi, \psi) + R^3 k_2(\phi, \psi) + R^2 k_3(\phi, \psi),$$

where $k_1(\phi, \psi) \geq 0$, $k_3(\phi, \psi) \geq 0$, $k_2(\phi, \psi) \in \mathbb{R}$ and it is allowed to be negative but if $k_1(\phi, \psi) = 0$, then $k_2(\phi, \psi) = 0$. If $k_1(\phi, \psi) = k_2(\phi, \psi) = 0$, then necessarily $k_3(\phi, \psi) \neq 0$. The k_j are continuous in $(\phi, \psi) \in \mathbb{T}$. When clear from the context, we write $k_j \equiv k_j(\phi, \psi)$.

Let $P_{\phi, \psi}(R) := R^2 k_1 + R k_2 + k_3$, so that $\tilde{V}_B(R\phi, R\psi) = R^2 P_{\phi, \psi}(R)$. Then $P_{\phi, \psi}(R)$ is a family of parabolas parameterized by $(\phi, \psi) \in \mathbb{T}$. Clearly $P_{\phi, \psi}(R) > 0$ for $R > 0$, otherwise the first part of the lemma is violated. Hence the desired property follows by noticing that for $R \geq 1$,

$$\min_{(\phi, \psi) \in \mathbb{T}} \tilde{V}_B(R\phi, R\psi) = R^2 \min_{(\phi, \psi) \in \mathbb{T}} P_{\phi, \psi}(R) \geq R^2 \min_{(\phi, \psi) \in \mathbb{T}} \mu(\phi, \psi),$$

where $\mu(\phi, \psi) := \min_{R \geq 1} P_{\phi, \psi}(R) > 0$ is a continuous function of \mathbb{T} . This completes the proof of the lemma.

In order to simplify out subsequent arguments, we translate the bosonic potential of the CSNW by a factor of \tilde{a} and write $V_B := \tilde{V}_B + \tilde{a}$. The conditions on V in the hypothesis of Theorem 2 below, ensure that this is completely harmless.

3. TRACE IDEALS AND THE BOSONIC HEAT KERNEL

In this section we show that $e^{-H_B t}$ has finite trace. We first review some standard mathematical facts regarding the theory of trace ideals and one-parameter semigroups, and provide rigorous definition of the operators H_B and H .

Let $1 \leq r < \infty$. We recall that a compact operator T is said to be in the r^{th} von Neumann-Schatten Class \mathcal{C}_r if, and only if,

$$\|T\|_r := \left(\sum_{n=0}^{\infty} \mu_n^{r/2} \right)^{1/r} < \infty$$

where μ_n are the singular values of T . The classes \mathcal{C}_1 and \mathcal{C}_2 are the operators of finite trace and of Hilbert-Schmidt type respectively, cf. [23]. If T is an integral operator acting on $L^2(\mathbb{R}^N; \mathbb{C}^d)$ with kernel

$K(x, y) \in \mathbb{C}^{d \times d}$, then

$$\|T\|_2^2 = \int_{(x,y) \in \mathbb{R} \times \mathbb{R}} \text{Tr} [K(x, y)K(x, y)^*] dx dy,$$

where here the asterisk denotes conjugated transpose of the corresponding matrix. Notice that $\|T\| \leq \|T\|_r$ for all $1 \leq r < \infty$. The sum of two operators in \mathcal{C}_r also lies in \mathcal{C}_r . In fact, the normed linear spaces $(\mathcal{C}_r, \|\cdot\|_r)$ are complete so these are complex Banach spaces. Furthermore \mathcal{C}_r are ideals with respect to the product in the algebra of all bounded linear operators and $\mathcal{C}_r \subset \mathcal{C}_s$ whenever $r < s$. See [24, ch.XI.9] or [23] for references to the proofs of these results and a more complete account on the elementary properties of trace ideals. In particular we will employ below the following well known facts.

- Duality: if $T \in \mathcal{C}_r$, $S \in \mathcal{C}_s$ and $q^{-1} = r^{-1} + s^{-1}$, where $q, r, s \geq 1$, then

$$(6) \quad \|TS\|_q \leq \|T\|_r \|S\|_s,$$

- Interpolation: if $T \in \mathcal{C}_2$ and $r > 2$, then

$$(7) \quad \|T\|_r \leq \|T\|_2^{2/r} \|T\|^{1-2/r}.$$

Notice that (6) ensures that any c_0 one-parameter semigroup lying on \mathcal{C}_r for some $r > 1$, should also lie on \mathcal{C}_1 .

Below and elsewhere we assume that N and d are fixed positive integers, and M is as in Lemma 1. In order to simplify the notation in our subsequent analysis, we rearrange the spacial coordinates and denote $(X, \mathcal{A}) \equiv x \in \mathbb{R}^N$.

Let $H_A := (-\Delta + c|x|^2) \otimes \mathbb{I}$ with domain

$$\text{Dom } H_A = H^2(\mathbb{R}^N; \mathbb{C}^d) \cap \widehat{H^2(\mathbb{R}^N; \mathbb{C}^d)} \subset L^2(\mathbb{R}^N; \mathbb{C}^d),$$

where $c > 0$ is a small constant. A reasoning involving the isometries U_α introduced below, easily shows that this operator is self-adjoint and positive in the above domain. The eigenvalues of H_A can be computed explicitly and $\|e^{-H_A t}\| < 1$ for all $t > 0$.

The heat kernel of H_A is found explicitly as follows. Let the unitary group of isometries $U_\alpha \phi(x) := e^{N\alpha/2} \phi(e^\alpha x)$, $\alpha \in \mathbb{R}$. Since

$$U_\alpha [(-\Delta + |x|^2) \otimes \mathbb{I}] U_{-\alpha} \phi(x) = [e^{-2\alpha} (-\Delta + e^{4\alpha} |x|^2) \otimes \mathbb{I}] \phi(x)$$

for all $\phi \in \text{Dom } H_A$ and

$$e^{-U_\alpha (P^2 + Q^2) U_{-\alpha} t} \phi(x) = U_\alpha e^{-(P^2 + Q^2) t} U_{-\alpha} \phi(x),$$

by virtue of Mehler's formula, (1),

$$K_t^A(x, y) = (w_1/\pi)^{N/2} \exp[2w_1(x \cdot y) - w_2(|x|^2 + |y|^2)] \mathbb{I},$$

$$w_1 = \frac{c^{1/2}\lambda}{1 - \lambda^2}, \quad w_2 = c^{1/2} \frac{1 + \lambda^2}{2(1 - \lambda^2)}, \quad \text{and} \quad \lambda = \exp[-2c^{1/2}t].$$

Moreover $e^{-H_A t}$ is Hilbert-Schmidt for all $t > 0$. Indeed,

$$\begin{aligned} \|e^{-H_A t}\|_2^2 &= \int_{\mathbb{R}^N \times \mathbb{R}^N} |K_t^A(x, y)|^2 dx dy \\ &= (w_1/\pi)^N \int \exp[-2(w_2^2 - w_1^2)|x|^2/w_2] \exp[-2w_2|y - (w_1/w_2)x|^2] dx dy \\ &= (w_1/\pi)^N \int \exp[-|x|^2/(2w_2)] dx \int \exp[-2w_2|y|^2] dy \\ &= w_1^N < \infty. \end{aligned}$$

In particular, $e^{-H_A t}$ is a compact operator so by virtue of the spectral mapping theorem, we can compute explicitly $\|e^{-H_A t}\| = e^{-Nc^{1/2}t}$. Notice that $Nc^{1/2}$ is the ground eigenvalue of H_A .

Let us now consider $H_B := (-\Delta + V_B) \otimes \mathbb{I}$. By virtue of lemma 1, $\text{Dom } H_B \subseteq \text{Dom } H_A$, then we may define rigorously the domain of H_B by means of Friedrichs extensions techniques, cf. [25]. In this domain H_B is self-adjoint. Furthermore, variational arguments along with (5), ensure that $H_B \geq 0$ and it has a complete set of eigenfunctions whose eigenvalues accumulate at $+\infty$.

By virtue of the Feynman-Kac formula (cf. [15]), by putting $c = M$ in the expression for H_A , we achieve

$$(8) \quad 0 < K_t^B(x, y) \leq K_t^A(x, y) \text{ for all } x, y \in \mathbb{R}^N \text{ and } t > 0.$$

Thus

$$(9) \quad \|e^{-H_B t}\|_2^2 = e^{-bt} \int_{\mathbb{R}^N \times \mathbb{R}^N} |K_t^B(x, y)|^2 dx dy \leq w_1^N,$$

so that $e^{-H_B t}$ is also a Hilbert-Schmidt operator and hence it has finite trace. Moreover, since $H_B \geq H_A$, the k^{th} eigenvalue of H_B is bounded below by the k^{th} eigenvalue of H_A and $\|e^{-H_B t}\| \leq e^{-NM^{1/2}t} < 1$ for all $t > 0$.

We conclude this section by examining the full hamiltonian H . According to [3, §3-5], H is a relatively bounded perturbation of H_B , $H := H_B + V_F$, where $V_F(x)$ is a Hermitian $d \times d$ matrix-valued function of \mathbb{R}^N linear in the variable x . Thus the (i, j) -th entry of V_F

satisfies

$$(10) \quad |V_F^{ij}(x)| \leq a(1 + |x|), \quad x \in \mathbb{R}^N,$$

for some constant a independent of i, j . This identity will play a crucial role in the description of the heat kernel of H . In physical terms, the potential V_F comprises the fermionic contribution in the CSNW model which, by the very special characteristics of this representation, happens to be dominated by the bosonic section of the theory.

By virtue of (5) and (10), the operator of multiplication by V_F is relatively bounded with respect to H_B with relative bound zero. Hence, the standard argument, see e.g. [26, Theorem 1.1, p.190], shows that H is a self-adjoint operator if we choose $\text{Dom } H := \text{Dom } H_B$.

In [3] a variational procedure is employed in order to show that H is bounded below, it possess a complete set of eigenfunctions and its spectrum comprises a discrete set of eigenvalues whose only accumulation point is $+\infty$. Notice that the spectral theorem together with Theorem 2 below, yields to an independent proof of this fact.

4. THE DYSON EXPANSION FOR e^{-Ht}

By employing a well known result due to Hille and Phillips, see [27, theorem 13.4.1], in this section we construct a series expansion for e^{-Ht} in terms of $e^{-H_B t}$ and V_F . We then prove that Lemma 1 and (10) ensure convergence of this series for all $t > 0$ in the topology of the von Neumann-Schatten class. Notice that Theorem 2, the main result of this section, may easily be formulated for general one-parameter semigroups arising from matrix integral operators. Nonetheless, in order to avoid distractions from our main purpose, we choose to restrict our notation to the concrete situation under discussion.

The following criterion on perturbation of generators of one-parameter semigroups may be found in [27, theorem 13.4.1], see also [28, §3.1]. Let $V : \mathbb{R}^N \longrightarrow \mathbb{C}^{d \times d}$ be a potential such that

$$(11) \quad \int_0^1 \|V e^{-H_B t}\| dt < \infty$$

where, by hypothesis, $e^{-H_B t} \phi$ lies in the domain of closure of V for all $\phi \in L^2(\mathbb{R}^N; \mathbb{C}^d)$ and $t > 0$. Then $H_B + V$ is the generator of a one-parameter semigroup $e^{-(H_B + V)t}$ for $t > 0$. Furthermore, the

operator-valued functions

$$\begin{aligned}
 (12) \quad & W_0(t)\phi := e^{-H_B t}\phi, \\
 & W_1(t)\phi := - \int_{s_1=0}^t e^{-H_B(t-s_1)} V e^{-H_B s_1} \phi \, ds_1, \\
 & W_2(t)\phi := \int_{s_1=0}^t \int_{s_2=0}^{s_1} e^{-H_B(t-s_1)} V e^{-H_B(s_1-s_2)} V e^{-H_B s_2} \phi \, ds_2 \, ds_1, \\
 & \vdots \\
 & W_k(t)\phi := (-1)^k \int_{s_1=0}^t \cdots \int_{s_k=0}^{s_{k-1}} e^{-H_B(t-s_1)} V e^{-H_B(s_1-s_2)} V \cdots \\
 & \quad \cdots e^{-H_B(s_{k-1}-s_k)} V e^{-H_B s_k} \phi \, ds_k \cdots ds_1,
 \end{aligned}$$

are well defined and there is a $\delta > 0$, small enough, such that

$$(13) \quad e^{-(H_B+V)t}\phi = \sum_{k=0}^{\infty} W_k(t)\phi$$

for all $0 < t < \delta$. Under hypothesis (11), the convergence of both the integrals and the series above is only guaranteed in the strong operator topology. This is Dyson's series for the heat kernel of a perturbed hamiltonian.

In order to recover the heat kernel of H , we may proceed as follows. Since $e^{-H_B t}$ is given by the kernel $K_t^B(x, y)$, a straightforward computation yields

$$W_k(t)\phi(x) = \int_{y \in \mathbb{R}^N} K_{t,k}(x, y)\phi(y) \, dy,$$

where $K_{t,0}(x, y) = K_t^B(x, y)$ and

$$K_{t,k}(x, y) := - \int_0^t \int_{\mathbb{R}^N} K_{t-s}^B(x, z) V(z) K_{s,k-1}(z, y) \, dz \, ds.$$

We remark that here $K_{t,k}(x, y)$ is a $d \times d$ matrix. The convergence in the strong operator topology ensures that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left[K_t(x, y) - \sum_{k=0}^n K_{t,k}(x, y) \right] \phi(y) \, dy = 0, \quad t < \delta,$$

for all $\phi \in L^2(\mathbb{R}^N; \mathbb{C}^d)$.

Although the above result may be useful in some applications, (11) only guarantees convergence in the strong operator topology for small $t > 0$. This provides a far from satisfactory description of $K_t(x, y)$. The following theorem shows that under more restrictive hypotheses on V ,

including those satisfied by V_F , uniform convergence is guaranteed for all $t > 0$ is the much stronger topology of the spaces \mathcal{C}_r .

Theorem 2. *Let $V : \mathbb{R}^N \longrightarrow \mathbb{C}^{d \times d}$ be a potential such that*

$$(14) \quad \max_{i,j=1,\dots,d} |V_{ij}(x)| \leq a(1 + |x|^\alpha) \quad x \in \mathbb{R}^N$$

for constants $a > 0$ and $0 \leq \alpha < 2$. Let $r > 2N/(2 - \alpha)$. Then $W_k(t) \in \mathcal{C}_r$ and $\sum_{k=0}^\infty \|W_k(t)\|_r < \infty$ for all $t > 0$. Hence

$$e^{-(H_B+V)t} = \sum_{k=0}^\infty W_k(t) \in \mathcal{C}_r,$$

where the series converges uniformly in the norm $\|\cdot\|_r$ for all $t > 0$.

Proof. Step 1: we first show that

$$(15) \quad \|Ve^{-H_B t}\|_r \leq bt^{-1+\varepsilon} \quad \text{for all } t > 0,$$

where $b > 0$ and $0 < \varepsilon < 1$ are constants independent of t . Below and elsewhere a_j are constants independent of t or n , but might depend on other parameters such as r , d , p or N .

For $n \in \mathbb{N}$, let

$$\Lambda_n = \{x \in \mathbb{R}^N : 2^n < |x| < 2^{n+1}\}$$

and denote by $\chi_n(x)$ the characteristic function of this set. Let $p > 1$ be a fixed parameter. By virtue of (8),

$$\begin{aligned} \|\chi_n e^{-H_B t}\|_2^2 &= \int_{\mathbb{R}^N \times \mathbb{R}^N} |\chi_n(x)|^2 |K_t^B(x, y)|^2 dx dy \\ &\leq a_1 w_1^N \int_{\Lambda_n} \exp[-|x|^2/(2w_2)] dx \int_{\mathbb{R}^N} \exp[-2w_2|y|^2] dy \\ &= a_2 \left(\frac{w_1}{w_2^{1/2}} \right)^N \int_{\Lambda_n} \exp[-|x|^2/(2w_2)] dx \\ &= a_3 \left(\frac{w_1}{w_2^{1/2}} \right)^N \int_{2^n}^{2^{n+1}} r^{N-1} \exp[-r^2/(2w_2)] dr \\ &\leq a_4 \left(\frac{w_1}{w_2^{1/2}} \right)^N \int_{2^n}^{2^{n+1}} (r/(2w_2)^{1/2})^{-(p+1)} r^{N-1} dr \\ &\leq a_5 w_1^N w_2^{(p+1-N)/2} 2^{-n(p+1-N)}. \end{aligned}$$

Notice that both w_1 and w_2 are continuous in t . Furthermore

$$\begin{aligned} w_1 &\sim t^{-1}, & w_2 &\sim t^{-1} & \text{as } t \rightarrow 0, \text{ and} \\ w_1 &\sim e^{-2M^{1/2}t}, & w_2 &\sim 1 & \text{as } t \rightarrow \infty. \end{aligned}$$

Also $\|\chi_n e^{-H_B t}\| \leq \|e^{-H_B t}\| \leq 1$ for all $t > 0$. Then, (7) yields

$$\|\chi_n e^{-H_B t}\|_r \leq a_6 t^{-(p+1+N)/(2r)} 2^{-n(p+1-N)/r}$$

for all $t > 0$.

Let $\tilde{V}(x) := \sum_{n=0}^{\infty} 2^{\alpha(n+1)} \chi_n(x)$. Then

$$\|\tilde{V} e^{-H_B t}\|_r \leq a_7 t^{-(p+1+N)/(2r)} \sum_{n=0}^{\infty} 2^{n[\alpha-(p+1-N)/r]}.$$

Thus, if r and p are large enough, such that

$$0 < \frac{p+1+N}{2r} < 1 \quad \text{and} \quad \frac{p+1-N}{r} > \alpha,$$

estimate (15) holds for \tilde{V} . This is possible only when $0 \leq \alpha < 2$, $r > 2N/(2-\alpha)$ and $p > \frac{\alpha+2}{2-\alpha}N - 1$, the first two conditions being precisely to the ones required in the hypothesis the theorem. By virtue of (7) and (9),

$$\|e^{-H_B t}\|_r \leq w_1^{N/r} \|e^{-H_B t}\|^{1-2/r} \leq a_8 t^{(\alpha-2)/\alpha}, \quad t > 0.$$

Then we may achieve (15) for V , by observing that

$$\begin{aligned} \|V e^{-H_B t}\|_r &= \|V(\tilde{V} + 1)^{-1}(\tilde{V} + 1)e^{-H_B t}\|_r \\ &\leq \|V(\tilde{V} + 1)^{-1}\| \|(\tilde{V} + 1)e^{-H_B t}\|_r \\ &\leq a_9 \sup_{x \in \mathbb{R}^N} \max_{i,j=1,\dots,d} \frac{|V_{i,j}(x)|}{\tilde{V}(x) + 1} (\|\tilde{V} e^{-H_B t}\|_r + \|e^{-H_B t}\|_r) \\ &\leq a_{10} (\|\tilde{V} e^{-H_B t}\|_r + \|e^{-H_B t}\|_r) \end{aligned}$$

for all $t > 0$.

From (15) it follows directly that (11) holds. Then the operators W_k in (12) are well defined and $e^{-(H_B+V)t}$ is given by expression (13) for $0 < t < \delta$.

Step 2: we show that $e^{-H_B t}$ is continuous for $t > 0$ in the norm $\|\cdot\|_r$. Let $\{\nu_1 \leq \nu_2 \leq \dots\} \subset (0, \infty)$ be the spectrum of H_B . Since $e^{-H_B t}$ is a compact operator, the spectrum of $e^{-H_B t}$ comprises the points $\{0\} \cup \{\dots \leq e^{-\nu_2 t} \leq e^{-\nu_1 t}\}$ and

$$\|e^{-H_B t}\|_r^r = \sum_{k=1}^{\infty} e^{-r\nu_k t} < \infty.$$

Since $e^{-H_B t}$ and $e^{-H_B s}$ have the same eigenfunctions, by virtue of the spectral mapping theorem and the dominated convergence theorem,

$$\|e^{-H_B t} - e^{-H_B s}\|_r^r = \sum_{k=1}^{\infty} |e^{-\nu_k t} - e^{-\nu_k s}|^r \rightarrow 0 \quad \text{as } |s - t| \rightarrow 0$$

so $e^{-H_B t}$ is continuous in $\|\cdot\|_r$.

Step 3: let us show that the conclusion of the theorem holds. In Section 2 we saw that the unperturbed semigroup $W_0(t) = e^{-H_B t} \in \mathcal{C}_1 \subset \mathcal{C}_r$ for all $t > 0$. In order to achieve the same conclusion for the remaining W_k we proceed as follows.

For $k = 1$, Step 2 ensures that $e^{-H_B(t-s_1)} V e^{-H_B s_1}$ is continuous in \mathcal{C}_r for $s_1 > 0$. Since

$$\begin{aligned} \int_{s_1=0}^t \|e^{-H_B(t-s_1)} V e^{-H_B s_1}\|_r ds_1 &\leq \int_{s_1=0}^t \|V e^{-H_B s_1}\|_r ds_1 \\ &< \frac{bt^\varepsilon}{\varepsilon} =: b(1) < \infty, \end{aligned}$$

it is also integrable in \mathcal{C}_r . Thus the (Riemann) integral in the definition of $W_1(t)$ converges in $\|\cdot\|_r$ and so $W_1(t) \in \mathcal{C}_r$ for all $t > 0$.

Let $c(k) = \int_0^1 (1-u)^{-1+\varepsilon} u^{-1+k\varepsilon} du$. For $k = 2$, notice that the \mathcal{C}_r -valued function $e^{-H_B(t-s)} V e^{-H_B(s-u)} V e^{-H_B u}$ is $\|\cdot\|_r$ -continuous in both s and u for all $0 < u < s < t$. Furthermore,

$$\begin{aligned} \int_{s_1=0}^t \int_{s_2=0}^{s_1} \|e^{-H_B(t-s_1)} V e^{-H_B(s_1-s_2)} V e^{-H_B s_2}\|_r ds_2 ds_1 &\leq \\ &\leq \int_{s_1=0}^t \int_{s_2=0}^{s_1} \|V e^{-H_B(s_1-s_2)}\|_r \|V e^{-H_B s_2}\|_r ds_2 ds_1 \\ &\leq b^2 \int_{s_1=0}^t \int_{s_2=0}^{s_1} (s_1 - s_2)^{-1+\varepsilon} s_2^{-1+\varepsilon} ds_2 ds_1 \\ &= b^2 \int_{s_1=0}^t s_1^{-1+2\varepsilon} ds_1 \int_{u=0}^1 (1-u)^{-1+\varepsilon} u^{-1+\varepsilon} du \\ &= \frac{b^2 t^{2\varepsilon}}{2\varepsilon} c(1) < \infty. \end{aligned}$$

Then, the integral in the definition of $W_2(t)$ also converges in the \mathcal{C}_r -norm and so $W_2(t) \in \mathcal{C}_r$ where

$$\|W_2(t)\|_r \leq \frac{b^2 t^{2\varepsilon}}{2\varepsilon} c(1)$$

for all $t > 0$.

Similar computations show that $W_k(t) \in \mathcal{C}_r$ and

$$\|W_k(t)\|_r \leq \frac{b^k t^{k\varepsilon}}{k\varepsilon} c(1)c(2) \cdots c(k-1) =: b(k)$$

for all $k = 3, 4, \dots$ and $t > 0$. Since $\sum_1^\infty b(k) < \infty$, the right hand side of (8) converges in the \mathcal{C}_q -norm for all $t > 0$. This completes the proof of Theorem 2.

In particular the conclusions of the above theorem hold when $V \equiv V_F$ so that H is the regularized hamiltonian of the CSNW model. Let $\{\lambda_1 \leq \lambda_2 \leq \dots\}$ be the set of eigenvalues of H . Since $e^{-Ht} \in \mathcal{C}_1$, $\sum_{k=1}^\infty e^{-\lambda_k t} < \infty$ for all $t > 0$. This remarkable result provides information about the asymptotic behaviour of λ_n as $n \rightarrow \infty$.

5. A FEYNMAN-KAC DESCRIPTION OF THE HEAT KERNEL

In this final section we discuss various aspects of the Feynman-Kac integral formulation of $K_t(x, y)$.

5.1. A formulation *via* standard Wiener measures. Let us first examine the standard line of reasoning that leads toward the formulation of a Feynman-Kac identity for the heat kernel of H , cf. [15]. For this we decompose $H = P^2 + V_S$, where $V_S(x) = V_B(x) \otimes \mathbb{I} + V_F(x)$, i.e. we regard the regularized hamiltonian as a perturbation of the free hamiltonian $P^2 \equiv -\Delta \otimes \mathbb{I}$. According to the results of [3], the eigenvalues of the overall potential matrix $V_S(x)$ are smooth functions of the variable of the configuration space, x , and they diverge to infinity as $x \rightarrow \infty$. This ensures that the potential operator is bounded from below. The Trotter-Kato formula, cf. [28] or [15], guarantees that

$$(16) \quad e^{-Ht}\phi(x) = \lim_{n \rightarrow \infty} \left(e^{-(\Delta \otimes \mathbb{I})t/n} e^{-V_S t/n} \right)^n \phi(x),$$

for all $\phi \in L^2(\mathbb{R}^N; \mathbb{C}^d)$. For finite $n \in \mathbb{N}$,

$$(17) \quad \text{Ker} \left[\left(e^{-(\Delta \otimes \mathbb{I})t/n} e^{-V_S t/n} \right)^n \right] = \int \prod_{j=1}^n e^{-(jt/n)V_S(x(jt/n))} dW_{x,y}^t,$$

cf. [15, Corollary 3.1.2], where the integrand is a matrix with bounded components so it is integrable. For each continuous path $x(\cdot)$ joining x and y ,

$$\lim_{n \rightarrow \infty} \prod_{j=1}^n e^{-(jt/n)V_S(x(jt/n))} =: \exp_{\text{ord}} \left[- \int_0^t V_S(x(s)) ds \right]$$

where, by definition, \exp_{ord} denotes the limit in the left hand side. The convergence of the finite difference scheme described in [29, Sections 3.1-3.5] guarantees the existence of this limit.

By virtue of the dominated convergence theorem for the Borel measure $W_{x,y}^t$,

$$\lim_{n \rightarrow \infty} \int \prod_{j=1}^n e^{-(jt/n)V_S(x(jt/n))} dW_{x,y}^t = \int \exp_{\text{ord}} \left[- \int_0^t V_S(x(s)) ds \right] dW_{x,y}^t.$$

Thus, identity (16) and the dominated convergence theorem, this time applied to the Lebesgue measure in \mathbb{R}^N , ensure that the left side of (17) approaches to $K_t(x, y)$ as $n \rightarrow \infty$. Thus we achieve the Feynman-Kac type identity

$$(18) \quad K_t(x, y) = \int \exp_{\text{ord}} \left[- \int_0^t V_S(x(s)) ds \right] dW_{x,y}^t.$$

Since all of the matrix functions involved are continuous, the above limits are guaranteed to exist for all x and y in \mathbb{R}^n .

5.2. A formulation *via* non-standard Wiener measures. We conclude this section by considering an alternative description of $K_t(x, y)$ based on the results of Section 4. Let V be a matrix-valued potential satisfying (14) and such that $H = H_B + V$ is bounded below. Without loss of generality we will assume that $H \geq cI$, for some $c > 0$, otherwise we may add a suitable constant to H and modify conveniently the resulting identity. Let

$$F(t) := e^{-(t/2)H_B} (\mathbb{I} - tV) e^{-(t/2)H_B}.$$

By virtue of (15), $F(t)$ is a bounded operator for all $t > 0$. It is also self-adjoint, $F(0) = I$ and $F(t)\phi \rightarrow \phi$ as $t \rightarrow 0$ for all ϕ . Notice that $F(t)$ is also a compact operator and, in fact, it lies in \mathcal{C}_r for $r > 2N/(2 - \alpha)$.

A direct calculation yields

$$(19) \quad \lim_{t \rightarrow 0} (t)^{-1} [F(t)\phi - \phi] = -H\phi$$

for all $\phi \in \text{Dom } H$.

Lemma 3. *There exists $t_0 > 0$ independent of ϕ , such that $0 < (\phi, F(t)\phi) < 1$ for all $t > t_0$ and $\phi \in L^2(\mathbb{R}^N; \mathbb{C}^d)$ with $\|\phi\| = 1$.*

Proof. Let ϕ be as in the hypothesis. By virtue of (15),

$$\|e^{-H_B t/2} V e^{-H_B t/2}\| \leq \|e^{-H_B t/2} V e^{-H_B t/2}\|_r \leq b t^{-1+\varepsilon} \quad \text{for all } t > 0,$$

where b is independent of t . Thus

$$(\phi, F(t)\phi) \geq \|e^{-H_B t} \phi\|^2 + t^\varepsilon \leq (1 + t^\varepsilon).$$

On the other hand, since $H \geq cI$,

$$(\phi, F(t)\phi) \leq (1 - tc)\|e^{-H_B t}\phi\|^2 \leq (1 - tc).$$

Hence the lemma is proven.

Since $F(t)$ is a self-adjoint operator, $\|F(t)\| < 1$ for all $t > t_0$. It is well known, cf. e.g. [28, Lemma 3.28], that (19) together with this condition ensure the validity of the following.

Corollary 4. *Let $t > 0$ be fixed. For all $\phi \in L^2(\mathbb{R}^N; \mathbb{C}^d)$,*

$$e^{-Ht}\phi(x) = \lim_{n \rightarrow \infty} F(t/n)^n \phi(x).$$

We can regard the latter corollary formally as a Feynman-Kac identity. Indeed, consider the measure $\tilde{W}_{x,y}^t$ in the space of continuous path constructed in the same fashion as the Wiener measure but using $K_t^B(x, y)$, instead of the kernel of the free evolution semigroup $e^{-P^2 t}$, for defining the measure of cylinder sets, cf. [15, p.49]. Then

$$\text{Ker}[F(t/n)^n](x, y) = \int \prod_{j=1}^n [\mathbb{I} - (t/n)V(x(jt/n))] d\tilde{W}_{x,y}^t.$$

In the limit $n \rightarrow \infty$, the integrand at the right hand side converges to $\exp_{\text{ord}} \left[- \int_0^t V(x(s)) ds \right]$. Hence

$$K_t(x, y) = \int \exp_{\text{ord}} \left[- \int_0^t V(x(s)) ds \right] d\tilde{W}_{x,y}^t.$$

Whenever $V \equiv V_F$, the latter, with the usual additional factor arising from the Fadeev-Popov procedure, is the path integral description of the regularized hamiltonian of the CSNW model. Notice that, although V_F is not bounded from below, it is dominated by the measure $\tilde{W}_{x,y}^t$.

CONCLUSION

We studied the quantization of the regularized hamiltonian of the compactified $D = 11$ supermembrane with non-zero central charge arising from a non-trivial winding. By showing that H is a relatively small perturbation of the bosonic hamiltonian, in Theorem 2 of Section 4 we provided a rigorous Dyson expansion for the heat kernel of this regularized hamiltonian. We demonstrated the convergence of this series in the topology of the von Neumann-Schatten class so that e^{-Ht} is of finite trace. These results are relevant in the analysis of the heat kernel of the $SU(N)$ regularized hamiltonian of the CSNW in the limit $N \rightarrow \infty$. The hypothesis of Theorem 2 suggests that the correct ideal to look at is \mathcal{C}_∞ , as N approaches ∞ . In Section 5 we discussed the

validity of the Feynman path integral description of the heat kernel. As a consequence of Corollary 4, we obtained a matrix Feynman-Kac formula.

The results established in this paper may be formulated in a completely abstract framework, so they may apply to other supersymmetric models. Nonetheless, our analysis does not include the important case of the supermembrane immersed on a $D = 11$ Minkowski space. This latter supermembrane has a positive hamiltonian but its potential is not bounded from below. It is rather unfortunate that, since this hamiltonian is not a perturbation of the bosonic contribution, it is still unclear whether a Feynman path integral formula exists for this case.

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